

Steady state entanglement of two coupled qubits

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The maximum entanglement between two coupled qubits in the steady state under two independent incoherent sources of excitation is reported. Asymmetric configurations where one qubit is excited while the other one dissipates the excitation are optimal for entanglement, reaching values three times larger than with thermal sources. The reason is the purification of the steady state mixture (that includes a Bell state) thanks to the saturation of the pumped qubit. Photon antibunching between the cross emission of the qubits can be used to experimentally evidence the large degrees of entanglement.

I. INTRODUCTION

A global state of a composite system is entangled when it cannot be written as a product of the states of the individual systems [1]. This is the basic quantum-mechanical property, with no classical analog, for quantum information technologies [2, 3]. Two coupled qubits, or two-level systems with ground $|g\rangle$ and excited $|e\rangle$ states, is the smallest and simplest composite system that can display entanglement. It is, therefore, the most suitable model to investigate its creation and processing as well as how environmental noise and decoherence brought by spontaneous decay and the external excitation affects it [4], which is a key point for quantum applications.

Two qubits can form four independent maximally entangled states, the so-called *Bell states*: $|\phi_{\pm}\rangle = (|gg\rangle \pm |ee\rangle)/\sqrt{2}$ and $|\psi_{\pm}\rangle = (|eg\rangle \pm |ge\rangle)/\sqrt{2}$. The last two are affected by a possible coupling between the qubits, and are also known in the atomic literature as the *symmetric* and *antisymmetric collective states* [5]. The formation and degradation of such states when subjected to spontaneous emission has been the object of much recent research [6–8], focusing on the preservation of entanglement into *decoherence-free subspaces* and taking advantage of the collective damping or effective coupling created between the qubits by interaction with common reservoirs [9–20].

The idea of environmentally induced entanglement has also been applied to the case where the two qubit interaction is mediated by a cavity mode (harmonic oscillator) which is excited by white noise (a thermal reservoir) [21], borrowing the idea from Ref. [22] where, on the contrary, entanglement is enhanced between two harmonic modes by mediation of a two-level system excited by white noise. In both cases, entanglement may survive in a steady state that is not the vacuum, but is very small ($< 0.4\%$). A two-level system has also been proposed as a mediator (or *coupler*), to build entanglement between qubits [23, 24].

Another possibility, close to the one addressed in the present text, is to consider two qubits already coupled, whose entanglement builds in the steady state despite dissipation and decoherence from two *independent* environments [25–33]. Entanglement, being essentially a property that requires

great purity of the state, is very sensitive to such decoherence. It is therefore important to look for optimization. Having a high degree of entanglement in the steady state means that it is robust and independent of the initial state, it remains stored forever in our system like a *quantum battery of entanglement* [32].

The coupling between the qubits can have different physical origins depending on the realization [3], e.g., Rydberg atoms couple through dipole-dipole interaction, weaker in the case of cold atoms [34], and so do excitons in single quantum dots or molecules [35–37]; superconducting qubits couple through mutual inductance [38]. Moreover, in all these implementations, the coupling can appear effectively through the virtual mediation of a coupler (a cavity or a wire mode) in the dispersive limit, in which case it is given by $g_{\text{eff}} \approx G_c^2/\Delta_c$, where G_c is the coupling of the qubits to the coupler and Δ_c the energy detuning to the qubits (considered much larger than the coupling) [39–41]. This scheme requires, for instance, placing the qubits into a cavity where the cavity mode acts as the coupler. One can take advantage of the QED techniques while obtaining an effective coupling essentially insensitive to the cavity decay and thermal fluctuations. The effective coupling between two Rydberg atoms through virtual photon exchange, while crossing a nonresonant cavity, was achieved in 2001 [42]. The final entangled state could be controlled by adjusting the atom-cavity detuning. A similar effective coupling was obtained between two superconducting qubits on opposite sides of a chip using microwave photons confined in a transmission line cavity [43]. The cavity was also used to perform multiplexed control and measurement of both qubit states. Effective coupling between two distant quantum dots embedded in a microcavity has also been recently achieved [44, 45].

Taking for granted that the two qubits are coupled, we center our attention on the situation where the qubits are also in contact with two independent excitation sources. Xu and Li [25] found that with two equally intense white-noise sources at the same temperature, no entanglement can be formed in the steady state. However, if only one qubit was subjected to a finite temperature source, some entanglement could be achieved. They also pointed out that the steady state entanglement exhibits a typical double *stochastic-resonance* as a function of the decoherence parameters of both qubits [22]. They found better but still small degrees of entanglement ($< 4\%$) and did not deepen on its origin, but their results show that an asymmetric flow of excitation through the qubits is beneficial for entanglement. Other authors, who did

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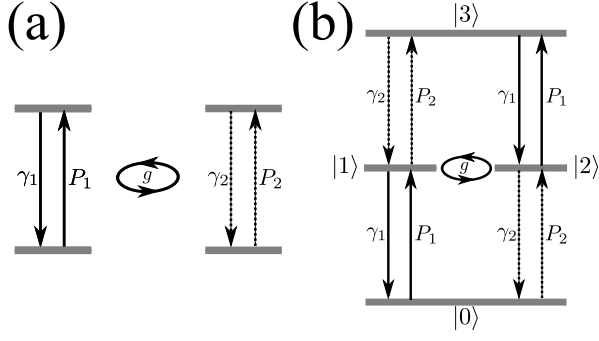


FIG. 1: Scheme of the two coupled qubits or two-level systems (a) and their energy levels (b), with coupling (g), pumping (P_i) and decay parameters (γ_i).

not consider unequal sources of excitation (having to recur to other mechanisms for entanglement generation), explored, on the other hand, configurations that are out of thermal equilibrium, where the excitation of the qubits is not necessarily produced by thermal sources but by more general processes able of inverting the qubits population [27, 28, 31]. Two qubits may undergo dissipation and pure dephasing but also an externally controllable and independent (in general) continuous pumping that can have great impact on the strong coupling reached in the steady state [46, 47].

In the present text, I put together different elements that have been addressed separately in previous studies of environmentally induced entanglement: direct coupling between the qubits and independent and different kinds of reservoirs that are not necessarily of a thermal nature. I give a complete picture of entanglement and its origin in the steady state of such a general system. As a result, I find a configuration where entanglement is significantly enhanced (31%), that is, more than three times as compared to the best thermal case and with a much better purity. I show that it can be evidenced by the antibunching of the two qubits cross emission.

The rest of this paper is organized as follows. In Sec. II, I introduce a theoretical model to describe two coupled qubits with decay, incoherent pumping and pure dephasing, and a quantity to quantify the degree of entanglement between them, the concurrence. In Sec. III, I discuss different entangled configurations and optimize the concurrence for the most suitable one: one qubit is excited while the other dissipates the excitation. This is compared with the thermal counterpart. In Sec. IV, I show how a strong antibunching between the two qubits emissions, is linked with high degrees of entanglement and I propose this effect as an indication of entanglement. In Sec. V, I study the effect of pure dephasing on entanglement. In Sec. VI, I present the conclusions.

II. THEORETICAL MODEL

Let us consider two qubits or two-level systems ($i = 1, 2$), with lowering operators σ_i , frequencies ω_i and coupled with strength g . Without loss of generality, we take the energy of the first qubit as a reference ($\omega_1 = 0$), from which the other

one is detuned by a small quantity $\Delta = \omega_1 - \omega_2$. The corresponding Hamiltonian reads:

$$H = -\Delta\sigma_2^\dagger\sigma_2 + g(\sigma_1^\dagger\sigma_2 + \sigma_2^\dagger\sigma_1), \quad (1)$$

Each qubit can be in the ground ($|g\rangle$) or excited ($|e\rangle$) states, whose direct product produces a Hilbert space of dimension 4 (see Fig. 1): $\{|0\rangle = |gg\rangle, |1\rangle = |eg\rangle, |2\rangle = |ge\rangle, |3\rangle = |ee\rangle\}$. The qubits are in contact with different kinds of environments, that provide or dissipate excitation (at rates P_i and γ_i , respectively) in an incoherent continuous way. Other interactions may purely bring dephasing to the coherent dynamics (at rates γ_i^d). These processes eventually drive any pure state into a statistical mixture of all possible states. A density matrix, ρ , properly describes the evolution of such a system. The general master equation we consider has the standard Liouvillian form [48]:

$$\partial_t \rho = i[\rho, H] + \sum_{i=1}^2 \left[\frac{\gamma_i}{2} \mathcal{L}_{\sigma_i} + \frac{P_i}{2} \mathcal{L}_{\sigma_i^\dagger} + \frac{\gamma_i^d}{2} \mathcal{L}_{\sigma_i^\dagger \sigma_i} \right] \rho, \quad (2)$$

with the corresponding Lindblad terms for the incoherent processes ($\mathcal{L}_O \rho \equiv 2O\rho O^\dagger - O^\dagger O \rho - \rho O^\dagger O$). If the two qubits shared a common environment, the Lindblad terms would share a single expression \mathcal{L}_J in terms of the collective operator: $J = \sigma_1 + \sigma_2$. Such collective terms are sources of entanglement, as explained in the introduction. In this text, I investigate the steady state of a system where they are not present, solving exactly the equation $\partial_t \rho = 0$.

In order to spell out the nature of the reservoirs that are in contact with the qubits, I express the pumping and decay rates in terms of new parameters Γ_i and r_i [49]:

$$\gamma_i = \Gamma_i(1 - r_i), \quad P_i = \Gamma_i r_i \quad (i = 1, 2). \quad (3)$$

The range $0 \leq r_i \leq 1$, includes a medium that only absorbs excitation (decay, $r_i = 0$), one which only provides it (pump, $r_i = 1$), as well as a the most common assumption of a thermal bath with finite temperature, (white noise, $r_i < 1/2$). The parameter $\Gamma_i = \gamma_i + P_i$ quantifies the interaction of each qubit with its reservoir as well as its effective spectral broadening.

Taking into account the evolution of the density matrix ρ of our bipartite system, one can conclude that in the steady state, it has the general block diagonal form:

$$\rho = \begin{pmatrix} \rho_{00} & 0 & 0 & 0 \\ 0 & \rho_{11} & \rho_{12} & 0 \\ 0 & \rho_{12}^* & \rho_{22} & 0 \\ 0 & 0 & 0 & \rho_{33} \end{pmatrix} \quad (4)$$

with

$$\rho_{00} = 1 - \langle n_1 \rangle - \langle n_2 \rangle + \langle n_1 n_2 \rangle, \quad (5a)$$

$$\rho_{ii} = \langle n_i \rangle - \langle n_1 n_2 \rangle, \quad i = 1, 2, \quad (5b)$$

$$\rho_{12} = \langle n_{12} \rangle^*, \quad (5c)$$

$$\rho_{33} = \langle n_1 n_2 \rangle, \quad (5d)$$

in terms of the operators $n_i = \sigma_i^\dagger \sigma_i$ and $n_{12} = \sigma_1^\dagger \sigma_2$. The average value $\langle n_i \rangle$ is the probability for qubit i to be excited,

regardless of the other one. $\langle n_{12} \rangle$ accounts for the population transfer between the qubits and $\langle n_1 n_2 \rangle$ for their effective coupling, in the sense that, independent qubits would lead to $\langle n_1 n_2 \rangle = \langle n_1 \rangle \langle n_2 \rangle$. The general expressions for the steady state of two coupled qubits can be written as $\langle n_i \rangle = P_i^{\text{eff}} / \Gamma_i^{\text{eff}}$, $\langle n_1 n_2 \rangle = (P_1 \langle n_2 \rangle + P_2 \langle n_1 \rangle) / (\Gamma_1 + \Gamma_2)$, $\langle n_{12} \rangle = \frac{2g(\langle n_1 \rangle - \langle n_2 \rangle)}{2\Delta + \Gamma_{\text{tot}}}$, where the effective parameters $P_i^{\text{eff}} = P_i + (P_1 + P_2)X_i$ and $\Gamma_i^{\text{eff}} = \Gamma_i + (\Gamma_1 + \Gamma_2)X_i$ are expressed in terms of an effective coherent exchange factor $X_i \equiv \frac{4g^2/\Gamma_{3i-1}}{\Gamma_{\text{tot}}[1 + (2\Delta/\Gamma_{\text{tot}})^2]}$ related to the Purcell rates [47]. X_i quantifies how efficiently the external inputs and outputs are distributed among the qubits thanks to the coherent coupling and despite the total decoherence, $\Gamma_{\text{tot}} = \Gamma_1 + \Gamma_2 + \gamma_1^d + \gamma_2^d$.

In Ref. [47], I showed that the steady state ρ of two coupled qubits is the same than that of a four-level system, that is, the system depicted in Fig. 1(b) with no correspondence to two two-level systems in Fig. 1(a), but rather to a single entity. This is the case of four single atomic levels or of a single quantum dot that can host two excitons and form a biexciton state. The results presented in the following sections are directly based on ρ or on averaged (single-time) quantities $\langle O \rangle$ computed as $\text{Tr}(\rho O)$ and, therefore, are also valid for a four-level system.

III. ENTANGLEMENT AND LINEAR ENTROPY

Among the four Bell states, $|\phi_{\pm}\rangle$ and $|\psi_{\pm}\rangle$, only the last two are achievable in the present configuration (since $\rho_{03} = 0$). Let us therefore write the most general entangled state that can be achieved in this system as $|\psi\rangle \equiv (|1\rangle + e^{i\beta}|2\rangle)/\sqrt{2}$. Logically, the larger the probability to find the qubits in such state (the closer ρ is to $|\psi\rangle\langle\psi|$), the larger is the degree of entanglement in the mixture represented by ρ . In order to make this statement mathematically precise, we can make explicit the entangled contribution to ρ by expressing it as

$$\rho = \rho_{00}|0\rangle\langle 0| + \rho_{33}|3\rangle\langle 3| + R_1|1\rangle\langle 1| + R_2|2\rangle\langle 2| + R_{\psi}|\psi\rangle\langle\psi| \quad (6)$$

where $R_1 = \rho_{11} - |\rho_{12}|$, $R_2 = \rho_{22} - |\rho_{12}|$ and $R_{\psi} = 2|\rho_{12}|$. R_i ($i = 1, 2, \psi$) are not probabilities (R_1, R_2 may be negative) but, when they are normalized as

$$\tilde{R}_i = \frac{|R_i|}{\rho_{00} + \rho_{33} + |R_1| + |R_2| + R_{\psi}}, \quad (7)$$

they represent the contribution of the pure states $|i\rangle$ to the mixture where the entangled state has been identified and set apart. In order to enhance entanglement, we must maximize \tilde{R}_{ψ} (ρ_{12}) while minimizing the populations ρ_{00} and ρ_{33} , and the differences \tilde{R}_1 and \tilde{R}_2 . The non-entangled contributions can be put together in a single expression to be minimized: $\tilde{R} = 1 - \tilde{R}_{\psi}$.

The degree of entanglement can be quantified by the *concurrence* (C) [50], which ranges from 0 (separable states) to 1 (maximally entangled states). It defined as $C \equiv$

$[\max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}]$, where $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are the eigenvalues in decreasing order of the matrix $\rho T \rho^* T$, with T being a anti diagonal matrix with elements $\{-1, 1, 1, -1\}$. The concurrence in this system is given by $C = 2\text{Max}\{0, |\rho_{12}| - \sqrt{\rho_{00}\rho_{33}}\}$, which shows a threshold behaviour that we anticipated above: the coherence between the intermediate states, $|\rho_{12}|$, must overcome the population of the spurious states ρ_{00}, ρ_{33} . A related important factor to build some concurrence, is the degree of purity in the system [51]. This is measured through the *linear entropy*, $S_L \equiv \frac{4}{3}[1 - \text{Tr}(\rho^2)]$, which is 0 for a pure state, and 1 for a maximally disordered state (where all four states occur with the same probability 1/4).

Without loss of generality, we can analyze the entanglement and linear entropy of our system in the steady state, by considering the parameters $\Delta \geq 0$, on the one hand, $0 \leq r_2 \leq r_1 \leq 1$ on the other hand, and arbitrary $\Gamma_1, \Gamma_2 \geq 0$. This simply implies that we label as 2 the qubit that is in contact with the medium which has the most dissipative nature. Let us ignore dephasing effects for the moment (we bring them back in Sec. V).

We start by noting that if $r_1 = r_2 = r$, that is, if the reservoirs are of the same nature, there is no entanglement in the system ($C = 0$), regardless of all the other parameters, since, in this case, the density matrix is diagonal with elements $\{(1-r)^2, r(1-r), r(1-r), r^2\}$, that is, a mixture of separable states. This result has already been pointed out in the literature [25, 27], however, let us insist on the fact that, it is not the amount of decoherence induced on the qubits by their environments what destroys entanglement, but their similarity in nature (or temperatures in the case of thermal baths). Let us then consider the cases with $r_2 < r_1$ in the rest of this section.

As in Ref. [51], I examine the region of the concurrence-linear entropy plane that our system can access in Fig. 2(a). The shaded region is reconstructed by randomly choosing all the parameters and computing their C and S_L . The accessible region is well below the black thin line for the *maximally entangled mixed states* [51], that provides the maximum concurrence achievable for a given linear entropy. More interestingly, the points are bounded in good approximation by a second (dashed blue) line specific to our system. This line corresponds to the extreme case of reservoirs with exactly opposite natures, $r_1 = 1$ and $r_2 = 0$, but equally strong influence on the qubits, $\Gamma_1 = \Gamma_2 = \Gamma$ (that is, $P_1 = \gamma_2 = \Gamma$, $P_2 = \gamma_1 = 0$). The steady state can be written in terms of a single unit-less complex number,

$$\alpha e^{i\beta} \equiv (\Delta - i\Gamma)/g, \quad (8)$$

with norm α and phase β : $\rho_{00} = \rho_{22} = \rho_{33} = \frac{1}{4+\alpha^2}$, $\rho_{11} = \frac{1+\alpha^2}{4+\alpha^2}$ and $\rho_{12} = \frac{\alpha e^{-i\beta}}{4+\alpha^2}$. The two qubits are sharing a single excitation $\langle n_1 \rangle + \langle n_2 \rangle = 1$. Note that both detuning (Δ) and the average decoherence (Γ) contribute symmetrically to α and have the same effect on the steady state: to make the coherent coupling less effective. The phase $\beta = -\arctan(\Gamma/\Delta)$, which is the same than that of the entangled state $|\psi\rangle$ formed in the steady state, can be rotated by changing these two parameters. This is a way to phase shift the entangled state obtained in the

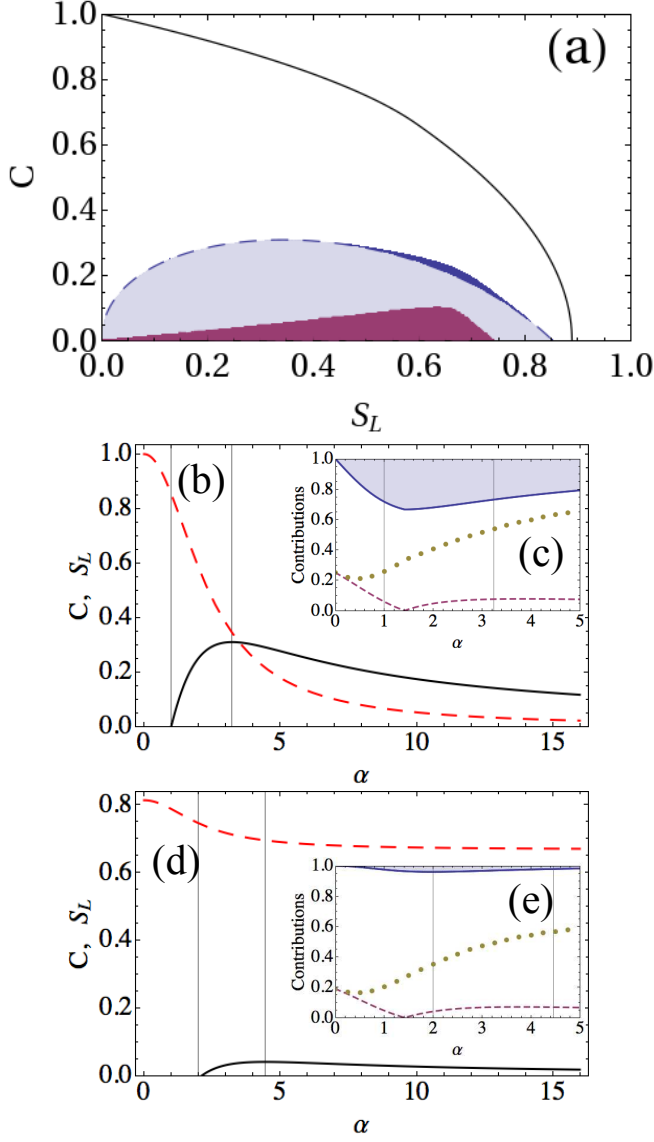


FIG. 2: (a) Distribution in the C - S_L plane of all the possible two qubit configurations (shaded region). The thin solid line corresponds to the maximum C for a given S_L in a general bipartite system. The dashed blue line corresponds to the optimal configuration ($r_1 = 1$, $r_2 = 0$, $\Gamma_1 = \Gamma_2 = \Gamma$, see Eq. (9a)), a good approximation to the maximal C vs S_L in our system (with the exception of the dark blue region above). Below, in dark purple, the particular case of thermal baths. (b) C (solid black) and S_L (dashed red) for the optimal case as a function of $\alpha = \sqrt{\Delta^2 + \Gamma^2}/g$. In inset (c), the non-entangled contributions to the steady state: \tilde{R}_1 (dotted brown), \tilde{R}_2 (dashed purple) and \tilde{R} (solid blue). The shaded area represents \tilde{R}_ψ (as $\tilde{R}_\psi + \tilde{R} = 1$). Idem in (d) and (e), but for two thermal baths at infinite and zero temperatures: $r_1 = 1/2$, $r_2 = 0$. The vertical guide lines mark the points where entanglement appears and where it is maximum, close to the points where \tilde{R} is minimum and \tilde{R} approaches \tilde{R}_1 , respectively.

system. Concurrence and linear entropy read

$$C = 2\text{Max}\left[\left\{0, \frac{\alpha - 1}{4 + \alpha^2}\right\}\right]$$

$$= \text{Max}\left[\left\{0, \frac{\sqrt{\frac{3S_L}{2}\left(1 - \frac{3S_L}{2}\right)} + \sqrt{1 - \frac{3S_L}{4}} - \frac{3S_L}{4}}{1 + \sqrt{1 - \frac{3S_L}{4}}}\right\}\right], \quad (9a)$$

$$S_L = \frac{16}{3} \frac{3 + \alpha^2}{(4 + \alpha^2)^2}. \quad (9b)$$

I plot them in Fig. 2(b), as a function of α moving leftwards along the dashed blue line of Fig. 2(a). The contributions \tilde{R}_1 (dotted brown), \tilde{R}_2 (dashed purple), \tilde{R} (solid blue) and \tilde{R}_ψ (shaded area) are also presented in the inset, Fig. 2(c), for a better understanding of the origin of entanglement. At vanishing α (or large effective coupling), the excitation is equally shared among the states and the system is maximally mixed (with $\tilde{R}_\psi = 0$). Concurrence becomes different from zero at $\alpha = 1$, which is close to the point where the non-entangled contribution to the density matrix, \tilde{R} reaches its minimum (and \tilde{R}_ψ its maximum). The contributions of the spurious states ρ_{00} , ρ_{33} have been considerably reduced while the coherence $|\rho_{12}|$ is sufficiently large to overcome them.

The maximum concurrence in the absolute for this system,

$$C_{\max} = (\sqrt{5} - 1)/4 \approx 31\%, \quad (10)$$

is reached at $\alpha = 1 + \sqrt{5}$. This is the region where a large contribution R_ψ is combined with low S_L . Moreover, the non-entangled contribution \tilde{R} becomes similar to \tilde{R}_1 meaning that the steady state is close to the mixture $M_\psi = R_\psi |\psi\rangle\langle\psi| + (1 - R_\psi) |1\rangle\langle 1|$, only between the entangled state $|\psi\rangle$ and $|1\rangle$. The large contribution of $|1\rangle$ to the steady state is expected since the first qubit is pumped and the second decays. What is less expected is that, by populating this state, we are purifying the total mixture and enhancing the presence of the entangled state. Increasing α further leads to the saturation of the system into state M_ψ and eventually to *self-quenching* of coherence [52]. Note, however, that concurrence decreases slowly and never becomes strictly zero again, due to the fact that $\rho \rightarrow M_\psi$ and, therefore, $C \rightarrow R_\psi$.

The small region in dark blue in Fig. 2(a), to the right and above the line in Eq. (9a), corresponds to cases more entangled for the same entropy, than the configuration previously discussed. Relaxing the previous conditions to $\Gamma_1 \neq \Gamma_2$, for instance, is enough to fill this area. In any case, configurations above the dashed line exist only for very mixed states, with $S_L > (17 - 3\sqrt{5})/30 \approx 0.34$, being less appealing for applications. In Fig. 3(a), I plot the corresponding concurrence as a function of Γ_1 and Γ_2 in order to show that it is robust to their difference: $C > 0$ as long as $\Gamma_1 + \Gamma_2 > 2$, and $C > C_{\max}/2 \approx 15\%$ in most of the area shown.

To conclude this analysis, one can check that if one medium provides an overall dissipation and the other one, an overall gain ($0 < r_2 < 1/2 < r_1 < 1$), then C can reach non-negligible values (above 10%). This is one of the important results in this text, the opposite nature of the reservoirs can lead the steady

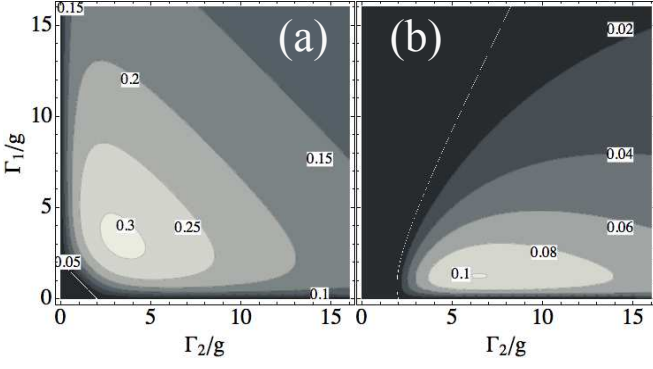


FIG. 3: Contour plots of concurrence C as a function of Γ_1 and Γ_2 for (a) $r_1 = 1, r_2 = 0$ and (b) $r_1 = 1/2, r_2 = 0$. To the left of the white lines, $C = 0$. The maximum value achieved with this system is in (a), $C_{\max} \approx 31\%$. In (b), with thermal baths, concurrence rises up to $C \approx 10\%$ for an asymmetric configuration.

state close to M_ψ , allowing for the highest degrees of entanglement in the system.

On the other hand, if we keep equal interactions with the reservoirs, $\Gamma_1 = \Gamma_2 = \Gamma$, but they are not restricted in their natures, the values of the concurrence decreases. Let us consider the case that has been previously studied in the literature, two thermal reservoirs in contact with two qubits. In our notation: $0 < r_2 < r_1 < 1/2$. In this case, the concurrence does not grow higher than 4% (as we said in the introduction), which is reached for the extreme case, $r_1 = 1/2, r_2 = 0$. Again, within the new restrictions, it is favorable for entanglement that excitation is provided through one qubit while the other only dissipates. The equivalent expressions to Eqs. (9) read:

$$C = \text{Max}\left[\left\{0, \frac{\alpha - \sqrt{9/4 + \alpha^2/2}}{4 + \alpha^2}\right\}\right], \quad (11a)$$

$$S_L = \frac{39 + 2\alpha^2(9 + \alpha^2)}{3(4 + \alpha^2)^2}. \quad (11b)$$

This case is featured in Fig. 2(d)-(e). We observe that C becomes different from zero again at the minimal \tilde{R} (maximal entangled contribution \tilde{R}_ψ) and that its maximum value is reached when \tilde{R}_1 approaches \tilde{R} . However, the concurrence remains one order of magnitude smaller than C_{\max} and the linear entropy does not drop to 0. With thermal excitation, \tilde{R}_ψ is always too small and the steady state is not close enough to M_ψ to exhibit a high degree of entanglement. However, in contrast with the optimally pumped case, one can increase entanglement from these figures by allowing $\Gamma_1 \neq \Gamma_2$. Concurrence is increased, filling the purple darker shaded region in Fig. 2(a). The maximum concurrence here is $C \approx 10\%$ at $\Gamma_1 \approx 1.24$ and $\Gamma_2 \approx 6.45$. This is shown in Fig. 3(b) where the highest values of C appear in light grey around those rates.

IV. ANTIBUNCHING

Is there an experimental observable that can evidence the high degrees of entanglement that we have analysed here?

One possibility is to reconstruct the steady state density matrix through quantum tomography, but this method involves complicated set ups and numerous and repeated measurements [53]. Here, I propose an alternative method that only involves photon counting, that is, the quantity:

$$\delta \equiv \langle n_1 \rangle \langle n_2 \rangle - \langle n_1 n_2 \rangle = \rho_{11}\rho_{22} - \rho_{00}\rho_{33}. \quad (12)$$

$\langle n_1 \rangle$ and $\langle n_2 \rangle$ are proportional to the intensity of the light emitted from each qubit, obtained by counting photons from each source, while $\langle n_1 n_2 \rangle$ is obtained by counting simultaneous photon detections. δ is directly linked to the second order cross correlation function [47] at zero delay, $g_{12}^{(2)}(0) = 1 - \delta / (\langle n_1 \rangle \langle n_2 \rangle)$. δ is zero if the qubits, acting as two random photon sources, are independent ($g_{12}^{(2)}(0) = g_{12}^{(2)}(\infty) = 1$), and different from zero if one qubit's emission is conditional to the other qubit's state. $\delta < 0$ implies that the simultaneous emission from both qubits is enhanced in the system as compared to the independent emissions. This is a necessary condition for photon *bunching*, although bunching also requires $g_{12}^{(2)}(0) > g_{12}^{(2)}(\tau)$. An example where $\delta < 0$, is the cross simultaneous emission of two coupled harmonic oscillators [46]. On the other hand, $\delta > 0$ implies that simultaneous emission from both qubits is less likely than in the uncoupled situation. Again, this is necessary for photon *antibunching* ($g_{12}^{(2)}(0) < g_{12}^{(2)}(\tau)$).

In the steady state of our system, the emission from one of the qubits is always antibunched ($g_{ii}^{(2)}(0) = 0 < g_{ii}^{(2)}(\tau)$, $i = 1, 2$, as it correspond to a two-level system) and the cross emission from both qubits fits $0 \leq \delta \leq 1/4$. One can check these limits from gathering δ s from many randomly generated configurations. It cannot go below zero because the only coherence and entanglement in the system come from the state $|\psi\rangle$ (that gives the maximum value $\delta = 1/4$) and not $|\phi\rangle \equiv (|0\rangle + e^{i\beta'}|3\rangle)/\sqrt{2}$ (that would give $\delta = -1/4$). The sign of δ is linked to the type of entangled state realised in the system, also when there is superposition or mixture with other states and $C < 1$. For instance, if we plot C versus δ for the maximally entangled mixed states [51] with entanglement provided by $|\psi\rangle$, we obtain the black thin line in Fig. 4, that abruptly falls at $\delta = 1/9$. This is because for this kind of states with $C < 2/3$, δ remains constant. One would obtain the symmetrical curve at negative δ , if the mixture was with $|\phi\rangle$. In this example, high degrees of entanglement are related to large δ . The other dotted lines appearing in Fig. 4 are more examples of this relationship between the type of entanglement and C with δ . The central black dotted line corresponds to the superposition or a mixture of $|\psi\rangle$ with $|\phi\rangle$, when $|\psi\rangle$ is the dominant state: $C = 4\delta$. There is a symmetric counterpart curve (not shown) in the opposite situation, where $|\phi\rangle$ is dominant, with $C = 4|\delta|$ and $\delta < 0$. The upper red dotted line corresponds to the superposition or mixture of $|\psi\rangle$ with $|0\rangle$ or with $|3\rangle$: $C = 2\sqrt{\delta}$. The counterpart curve, with $|\phi\rangle$, is symmetrical. The space between these two dotted lines could be filled with mixtures of $|\psi\rangle$ with both $|0\rangle$ and $|3\rangle$. The lower blue dotted line corresponds to the mixture of $|\psi\rangle$ with $|1\rangle$, the state M_ψ : $C = 1 - \sqrt{1 - 4\delta}$. In all these cases, large δ is correlated with large C , although the connexion is rather

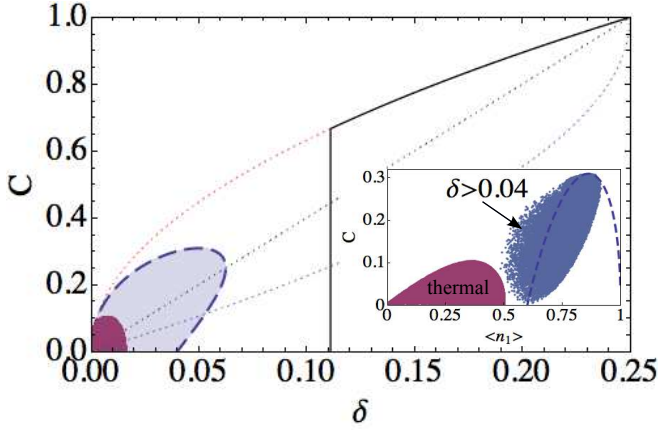


FIG. 4: Distribution in the C - δ plane of all the possible qubit configurations (shaded region). The thin solid and dotted lines correspond to different examples of entangled mixed states, with $|\psi\rangle$ (see the main text). The pure state $|\psi\rangle$ corresponds to the extreme point $(1/4, 1)$. The dashed blue line corresponds to the configuration $(r_1 = 1, r_2 = 0, \Gamma_1 = \Gamma_2 = \Gamma)$, increasing Γ anti-clockwise) which encloses all possible realizations. Inside, in dark purple, the particular case of thermal reservoirs. In inset: C vs $\langle n_1 \rangle$ for thermal reservoirs (where $\langle n_1 \rangle < 0.5$, in dark purple) and for those configurations where $\delta > 0.04$ (in blue). The dashed blue line ($r_1 = 1, r_2 = 0, \Gamma_1 = \Gamma_2 = \Gamma$) goes clockwise with increasing Γ .

general and not exclusive enough to define any *entanglement witness* in terms of δ .

Let us go back to our system and investigate how to use these correlations to extract information about C from the measured δ . The shaded region in Fig. 4 corresponds, as in Fig. 2, to the situations realised in our system. It is completely enclosed this time by the dashed blue line, which corresponds to reservoirs with opposite natures (as analysed in the previous section). In this limiting case, δ reads

$$\delta = \left(\frac{\alpha}{4 + \alpha^2} \right)^2. \quad (13)$$

Thanks to this analytical boundary, we can turn the general statement that there is some correlation between δ and C into a more accurate (mathematical) one: $C_-(\delta) \leq C \leq C_+(\delta)$ where

$$C_{\pm}(\delta) = \frac{2\sqrt{2\delta}\sqrt{1 \pm \sqrt{1 - 16\delta}} - 8\delta - \sqrt{2\delta}}{1 \pm \sqrt{1 - 16\delta}}. \quad (14)$$

These inequalities become most stringent when $\delta > 0.04$, for instance $\delta > 0.061$ implies $20\% < C < 28.3\%$. More precise information can be obtained if $\langle n_1 \rangle$ is included in the analysis, looking at the inset of Fig. 4. In blue, we see a cloud of numerically generated points where $\delta > 0.04$. There is also a clear correlation between large, unsaturated, population of the dot ($0.8 < \langle n_1 \rangle < 0.91$) and large degrees of entanglement ($10\% < C < C_{\max}$).

Such large δ and C , cannot be obtained with thermal reservoirs for the qubits. The small accessible area in that case, is shaded in darker purple in Fig. 4 (within the dashed blue

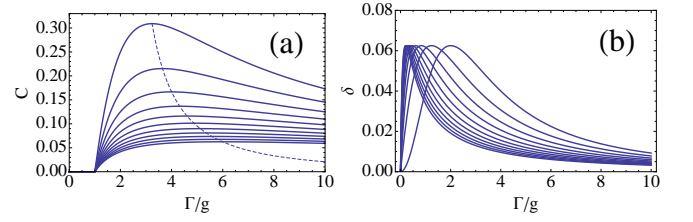


FIG. 5: Effect of dephasing on the results for opposite reservoirs ($r_1 = 1, r_2 = 0, \Gamma_1 = \Gamma_2 = \Gamma$), of C (a) and δ (b) as a function of Γ . The set of curves corresponds to values of γ^d from 0 to $20g$, increasing in steps of $2g$. Entanglement (a) is diminished by pure dephasing, going from top to bottom curves, and the maximum is reached at higher Γ s (see the dashed line, joining the maxima of all curves). The maximum δ remains $1/16$ for all values of dephasing although this is reached for lower Γ s.

curve) and in the inset. One cannot use δ or $\langle n_1 \rangle$ as entanglement indicators because, for all δ or $\langle n_1 \rangle$, C can take a broad range of values that always includes 0.

V. PURE DEPHASING

Pure dephasing provides extra decay for the coherence in the system. It weakens the correlations established between the qubits and is, therefore, an enemy of entanglement. We can see this in Fig. 5(a) where I plot the effect of increasing dephasing ($\gamma_1^d = \gamma_2^d = \gamma^d$) on entanglement. The curve at the top is the same as in Fig. 2(b), with opposite kinds of reservoirs. The rest of the curves correspond to increasing values of the dephasing rate in steps of $2g$ up to $\gamma^d = 20g$. Entanglement decreases, and its maximum value for a given γ^d , requires higher Γ . The set of maximum C and the corresponding required Γ are plotted with a dashed thin line superimposed to the curves for clarity. Entanglement is quite robust in this configuration, it disappears but asymptotically and very slowly. Note that γ^d must be one order of magnitude larger than g so that C is decreased to the values obtained with thermal reservoirs (10%).

In Fig. 5(b), I plot the counterpart curves for δ , that shrink and move leftwards with dephasing. However, the maximum δ remains $1/16$ for all dephasing, taking place at lower Γ s. Given that the tendency of the maximum δ is the opposite to that of the maximum C , the possibility of using δ as an indicator of entanglement fails at large dephasing.

VI. CONCLUSIONS

I have computed the entanglement (C) and linear entropy (S_L) for two coupled qubits in the steady state created by an incoherent continuous excitation. I have studied, not only the case where the excitation is of a thermal origin, but also a more general out-of-equilibrium situation where populations can be inverted ($\langle n_1 \rangle > 0.5$). In this case, I find that entanglement can be greatly enhanced (C up to 31%) as compared to the best thermal values (C up to 10%), with also a much

higher purity of the state. This is obtained in a configuration where one qubit essentially dissipates excitation while the other essentially gains it. I have used both numerical results, in the most general case, and analytical formula, in this optimal case, to fully understand and characterise entanglement formation. Entanglement (provided by the entangled state $|\psi\rangle$) is enhanced in the steady state when the pumped qubit approaches saturation, because this removes population

from spurious states ($|0\rangle$ and $|3\rangle$) and purifies the statistical mixture, even in the presence of pure dephasing. Finally, I have shown that the quantity $\delta = \langle n_1 \rangle \langle n_2 \rangle - \langle n_1 n_2 \rangle$, that can be measured experimentally by photon counting, can be used as an indicator of high degrees of entanglement in this system and specially for the optimal configuration.

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